

A GAUSSIAN MEASURE OF QUANTUM PHASE NOISE

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ABSTRACT

We study the width of the semiclassical phase distribution of a quantum state in its dependence on the average number of photons $\langle m \rangle$ in this state. As a measure of phase noise, we choose the width $\Delta\phi$ of the best Gaussian approximation to the dominant peak of this probability curve. For a coherent state this width decreases with the square root of $\langle m \rangle$, whereas for a truncated phase state it decreases linearly with increasing $\langle m \rangle$. For an optimal phase state, $\Delta\phi$ decreases exponentially — but so does the area “caught” underneath the peak: All the probability is stored in the broad wings of the distribution.

I. INTRODUCTION

The ultimate quantum limit in the goal of optically detecting gravitational waves is to operate a Michelson interferometer with light in a quantum state that minimizes the phase noise at a given mean number of photons (Ref. 1). But what is a good measure for phase noise? Should we consider the inverse peak height of the probability distribution (Ref. 2) — the so-called reciprocal likelihood — or perhaps the second moment of the phase distribution (Ref. 3) or even the periodic measure advocated in Refs. 4 and 5? These are all based on the idea of a phase distribution — but we recall that this in itself is not a trivial construction since the concept of a Hermitian phase operator is not without complications (Ref. 6).

In the present paper we therefore start from the *semiclassical* phase distribution $W_\phi[|\psi\rangle]$ of a quantum state $|\psi\rangle$ (Refs. 7 and 8). We consider states that have a single pronounced maximum of W_ϕ at a phase value that, without loss of generality, we take at $\phi = 0$. We approximate (Ref. 9) this peak by a Gaussian distribution with an identical height of $W_{\phi=0}$. The distribution's width $\Delta\phi$ is determined by the curvature of W_ϕ at $\phi=0$, together with $W_{\phi=0}$. The examples of a coherent state $|\psi_{\text{coh}}\rangle$ and a truncated phase state $|\psi_p\rangle$ will illustrate this scheme: Their widths $\Delta\phi[|\psi_{\text{coh}}\rangle]$ and $\Delta\phi[|\psi_p\rangle]$ decrease as the *square root* and *linearly* with increasing average photon number $\langle m \rangle$, respectively. In the case of the optimal phase state — the state (Ref. 2) that minimizes the reciprocal likelihood — we find (Ref. 9) even an *exponential* decay for $\Delta\phi$. However, in contrast to the coherent or the truncated phase state, the area underneath this maximum is not normalized but decays as well (Ref. 10).

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II. FROM PHASE FUNCTIONAL TO GAUSSIAN-APPROXIMATED PHASE DISTRIBUTION

In this section we start from the semiclassical phase amplitude functional $w_\phi[|\psi\rangle]$ of a quantum state $|\psi\rangle$ and derive a Gaussian approximation to the dominant maximum of this probability curve.

In the semiclassical limit the phase distribution $W_\phi[|\psi\rangle]$ of a quantum state

$$|\psi\rangle = \mathcal{N} \sum_{m=0}^{\infty} \psi_m |m\rangle \quad (2.1)$$

represented as a superposition of photon number states $|m\rangle$, with expansion coefficients $\psi_m \equiv \langle m | \psi \rangle$ and the normalization \mathcal{N} such that $\langle \psi | \psi \rangle = 1$, follows from the phase functional

$$w_\phi[|\psi\rangle] \equiv \frac{\mathcal{N}}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \psi_m e^{im\phi} \quad (2.2a)$$

via

$$\begin{aligned} W_\phi[|\psi\rangle] &\equiv \left| w_\phi[|\psi\rangle] \right|^2 \\ &= \frac{\mathcal{N}^2}{2\pi} \sum_{m,n=0}^{\infty} \psi_m \psi_n e^{i(m-n)\phi}. \end{aligned} \quad (2.2b)$$

For the sake of simplicity, we consider in this article only quantum states such that $\psi_m^* \geq 0$. Hence, for the phase value $\phi = 0$ the terms in the sum of Eq. (2.2b) add constructively, whereas for $\phi \neq 0$ cancellations occur. This results in a maximum at $\phi = 0$.

An approximate analytical expression for $W_\phi[|\psi\rangle]$ in the neighborhood of the origin follows from an expansion of W_ϕ into a Taylor series around $\phi = 0$, that is

$$\begin{aligned} W_\phi &= W_{\phi=0} + \frac{1}{2} W''_{\phi=0} \phi^2 + \dots \\ &\equiv W_{\phi=0} \left\{ 1 + \left[W''_{\phi=0} / (2W_{\phi=0}) \right] \phi^2 \right\}. \end{aligned} \quad (2.3)$$

Here primes denote differentiation with respect to ϕ and we have used the property

$$W'_{\phi=0} = \frac{\mathcal{N}^2}{2\pi} \sum_{m,n=0}^{\infty} i(m-n) \psi_m \psi_n = 0,$$

following from Eq. (2.2b). With the help of Eq. (2.3), we arrive at

$$W_\phi^{(\text{peak})} = \exp(\ln W_\phi) \equiv W_{\phi=0} \exp \left[- \left(\frac{\phi}{\Delta\phi} \right)^2 \right] \quad (2.4a)$$

where the width $\Delta\phi$ of this Gaussian is given by

$$\begin{aligned} \Delta\phi^2 &\equiv \frac{2W_{\phi=0}}{\left| W''_{\phi=0} \right|} \\ &= \frac{\left(\sum_{m=0}^{\infty} \psi_m \right)^2}{\left(\sum_{m=0}^{\infty} m^2 \psi_m \right) \left(\sum_{m=0}^{\infty} \psi_m \right) - \left(\sum_{m=0}^{\infty} m \psi_m \right)^2}. \end{aligned} \quad (2.4b)$$

We emphasize that this procedure is valid for any state whose phase probability $W_\phi[|\psi\rangle]$ enjoys a maximum at $\phi = 0$.

The area underneath this Gaussian-approximated peak reads as

$$A_{\text{Gauss}} = \int_{-\infty}^{\infty} d\varphi W_{\varphi=0} e^{-(\varphi/\Delta\varphi)^2}$$

$$= \sqrt{\pi} \Delta\varphi W_{\varphi=0} \quad (2.5)$$

This clearly demonstrates that the Gaussian fit of the peak, Eq. (2.4), is different from the properly normalized Gaussian $G(\varphi) = \pi^{-1/2}$

$(\Delta\varphi)^{-1} \exp [-(\varphi/\Delta\varphi)^2]$. Whereas $W_{\varphi}^{(\text{peak})}$ is tailored to have a height identical to $W_{\varphi=0} [|\psi\rangle]$, the height of G , that is, $\pi^{-1/2} (\Delta\varphi)^{-1}$, adjusts itself to the width of the Gaussian, so as to keep the area normalized.

III. PHASE NOISE AND AVERAGE NUMBER OF PHOTONS

In this section we apply the Gaussian approximation, Eq. (2.4a), in order to discuss the width $\Delta\varphi$, Eq. (2.4b) of the phase distribution of:

(i) a coherent state of large displacement $\alpha \gg 1$,

$$|\psi_{\text{coh}}\rangle = (2\pi)^{-1/4} \alpha^{-1/2} \sum_{m=0}^{\infty} e^{-\left(\frac{m-\alpha}{2\alpha}\right)^2} |m\rangle, \quad (3.1)$$

(ii) a truncated phase state

$$|\psi_p\rangle = (m_0+1)^{-1/2} \sum_{m=0}^{m_0} |m\rangle, \quad (3.2)$$

and

(iii) the optimal phase state

$$|\psi_s\rangle = \mathcal{N} \sum_{m=0}^{m_0} \frac{1}{1+m} |m\rangle \quad (3.3)$$

recently proposed in Ref. 2.

These states are normalized to unity. For a coherent state, Eq. (3.1), we arrive at

$$\langle\psi_{\text{coh}}|\psi_{\text{coh}}\rangle = (2\pi)^{-1/2} \alpha^{-1} \sum_{m=0}^{\infty} e^{-\left(\frac{m-\alpha}{\sqrt{2}\alpha}\right)^2}$$

$$\equiv \pi^{-1/2} \int_{-\infty}^{\infty} d\bar{m} e^{-\bar{m}^2}$$

$$= 1 \quad (3.4)$$

where we have replaced the summation by an integration. For the truncated phase state $|\psi_p\rangle$ we find directly

$$\langle\psi_p|\psi_p\rangle = (m_0+1)^{-1} \sum_{m=0}^{m_0} 1 = 1,$$

whereas for the optimal state $|\psi_s\rangle$, Eq. (3.3), the normalization constant \mathcal{N} is given implicitly by

$$\langle\psi_s|\psi_s\rangle = \mathcal{N}^2 \sum_{m=0}^{m_0} (1+m)^{-2}$$

$$= \mathcal{N}^2 \left\{ \sum_{k=1}^{\infty} k^{-2} - \sum_{k=0}^{\infty} (m_0+2+k)^{-2} \right\},$$

that is, (Ref. 11)

$$1 = \mathcal{N}^2 \left\{ \frac{\pi^2}{6} - \zeta(2; m_o + 2) \right\}$$

$$= \mathcal{N}^2 \left\{ \frac{\pi^2}{6} - \frac{1}{m_o + 1} + O(m_o^{-2}) \right\}. \quad (3.5)$$

Here,

$$\zeta(s, v) \equiv \sum_{k=0}^{\infty} (v+k)^{-s}$$

denotes the generalized Riemann zeta function (Ref. 12).

A. Coherent State

For the coherent state, Eq. (3.1), the expansion coefficients ψ_m read

$$\psi_m = (2\pi)^{-1/4} \alpha^{-1/2} \exp \left[- \left(\frac{m - \alpha^2}{2\alpha} \right)^2 \right]$$

and the width $\Delta\phi$, given by Eq. (2.4b) reduces to

$$\Delta\phi^2 [|\psi_{\text{coh}}\rangle] \equiv \frac{\left(\int_{-\infty}^{\infty} dm \psi_m \right)^2}{\left(\int_{-\infty}^{\infty} dm m^2 \psi_m \right) \left(\int_{-\infty}^{\infty} dm \psi_m \right) - \left(\int_{-\infty}^{\infty} dm m \psi_m \right)^2}. \quad (3.6)$$

Here, we have once again replaced summations over m by integrations. We evaluate the Gaussian integrals most economically by applying the symmetry of ψ_m with respect to α^2 before performing the integrals. This yields

$$\int_{-\infty}^{\infty} dm m^2 \psi_m = \int_{-\infty}^{\infty} dm (m - \alpha^2)^2 \psi_m + 2\alpha^2 \int_{-\infty}^{\infty} dm (m - \alpha^2) \psi_m + \alpha^4 \int_{-\infty}^{\infty} dm \psi_m = \int_{-\infty}^{\infty} dm (m - \alpha^2)^2 \psi_m + \alpha^4 \int_{-\infty}^{\infty} dm \psi_m$$

and

$$\int_{-\infty}^{\infty} dm m \psi_m = \int_{-\infty}^{\infty} dm (m - \alpha^2) \psi_m + \alpha^2 \int_{-\infty}^{\infty} dm \psi_m = \alpha^2 \int_{-\infty}^{\infty} dm \psi_m.$$

This result reduces Eq. (3.6) to

$$\begin{aligned}
\Delta\phi^2 [|\psi_{\text{coh}}\rangle] &= \frac{\int_{-\infty}^{\infty} dm \psi_m}{\int_{-\infty}^{\infty} dm (m-\alpha^2)^2 \psi_m} \\
&= \frac{\int_{-\infty}^{\infty} dy \exp\left[-\left(\frac{y}{2\alpha}\right)^2\right]}{\int_{-\infty}^{\infty} dy y^2 \exp\left[-\left(\frac{y}{2\alpha}\right)^2\right]} \\
&= \frac{1}{2\alpha^2} \quad (3.7a)
\end{aligned}$$

The average number of photons $\langle m \rangle$ follows from the normalization condition, Eq. (3.4), as

$$\begin{aligned}
\langle m \rangle &\equiv \int_{-\infty}^{\infty} dm m \psi_m^2 \\
&= \int_{-\infty}^{\infty} dm (m-\alpha^2) \psi_m^2 + \alpha^2 \int_{-\infty}^{\infty} dm \psi_m^2 \\
&= \alpha^2
\end{aligned}$$

and hence

$$\Delta\phi [|\psi_{\text{coh}}\rangle] \equiv \frac{1}{\sqrt{2 \langle m \rangle}} \quad (3.7b)$$

for large $\langle m \rangle$. Asymptotically, the width $\Delta\phi [|\psi_{\text{coh}}\rangle]$ of the phase distribution of a coherent state decreases inversely as the square root of the average photon number.

We now evaluate the area underneath the Gaussian approximation, Eq. (2.5), for the coherent state, Eq. (3.1). The maximum value of the phase distribution reads

$$\begin{aligned}
W_{\phi=0} [|\psi_{\text{coh}}\rangle] &= \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} dm \psi_m \right)^2 \\
&= \sqrt{\frac{2}{\pi}} \alpha \\
&= \sqrt{\frac{2}{\pi} \langle m \rangle} \quad (3.8)
\end{aligned}$$

and hence the area of the Gaussian is

$$\begin{aligned}
A_{\text{Gauss}} [|\psi_{\text{coh}}\rangle] &= \sqrt{\pi} \cdot \Delta\phi [|\psi_{\text{coh}}\rangle] \\
&\quad \cdot W_{\phi=0} [|\psi_{\text{coh}}\rangle] = 1 \quad (3.9)
\end{aligned}$$

Thus the Gaussian approximation, Eq. (2.4) for a coherent state is properly normalized. Its width $\Delta\phi [|\psi_{\text{coh}}\rangle]$ decreases linearly with α but its height $W_{\phi=0} [|\psi_{\text{coh}}\rangle]$ increases linearly with $\alpha \sim \sqrt{\langle m \rangle}$ to leave the area A_{Gauss} constant.

B. Truncated Phase State

We now turn to the discussion of the truncated phase state $|\psi_p\rangle$, Eq. (3.2). According to this equation, the expansion coefficients ψ_m read

$$\psi_m = \begin{cases} 1, & \text{for } 0 \leq m \leq m_0 \\ 0, & \text{for } m > m_0 \end{cases}$$

which reduces $\Delta\phi [|\psi_p\rangle]$ to

$$\begin{aligned}
\Delta\varphi^2 [|\psi_p\rangle] &= \frac{\left(\sum_{m=0}^{m_0} 1\right)^2}{\left(\sum_{m=0}^{m_0} m^2\right)\left(\sum_{m=0}^{m_0} 1\right) - \left(\sum_{m=0}^{m_0} m\right)^2} \\
&= \frac{(m_0 + 1)^2}{\frac{1}{6} m_0 (m_0 + 1)^2 (2m_0 + 1) - \frac{1}{4} m_0^2 (m_0 + 1)^2} \\
&= \frac{12}{m_0 (m_0 + 2)} \\
&= \frac{12}{m_0^2} + O(m_0^{-3}). \quad (3.10)
\end{aligned}$$

Here, we have made use of the summation formulae (Ref. 11)

$$\sum_{m=1}^{m_0} 1 = m_0,$$

$$\sum_{m=1}^{m_0} m = \frac{1}{2} m_0 (m_0 + 1),$$

and

$$\sum_{m=1}^{m_0} m^2 = \frac{1}{6} m_0 (m_0 + 1) (2m_0 + 1). \quad (3.11)$$

When we express m_0 in terms of the average number of photons

$$\langle m \rangle = (1 + m_0)^{-1} \sum_{m=1}^{m_0} m = \frac{1}{2} m_0,$$

we find

$$\Delta\varphi [|\psi_p\rangle] \cong \frac{\sqrt{3}}{\langle m \rangle}, \quad (3.12)$$

again for large $\langle m \rangle$. The width $\Delta\varphi [|\psi_p\rangle]$ of the phase distribution of the truncated phase state decreases linearly with the average photon number $\langle m \rangle$.

The value of W_φ at $\varphi=0$ reads

$$\begin{aligned}
W_{\varphi=0} &= (2\pi)^{-1} (m_0 + 1)^{-1} \left(\sum_{m=0}^{m_0} 1\right)^2 \\
&= (2\pi)^{-1} (m_0 + 1) \\
&\cong \frac{1}{\pi} \langle m \rangle, \quad (3.13)
\end{aligned}$$

and yields for the area underneath the Gaussian approximation, Eq. (2.4), of the truncated phase state, Eq. (3.2),

$$\begin{aligned}
A_{\text{Gauss}} [|\psi_p\rangle] &= \sqrt{\pi} \cdot \Delta\varphi [|\psi_p\rangle] \cdot W_{\varphi=0} [|\psi_p\rangle] \\
&= \sqrt{\frac{3}{\pi}} \cong 0.98, \quad (3.14)
\end{aligned}$$

i.e., *almost* perfect normalization. Again, we note from Eqs. (3.10) and (3.13) that the width $\Delta\varphi$ decreases with m_0 in the same manner as the maximum height $W_{\varphi=0}$ increases — keeping the area A_{Gauss} normalized and, more importantly, keeping it independent of m_0 .

C. Optimal Phase State

We conclude our illustration of the Gaussian-approximated phase distribution by applying it to the optimal phase state $|\psi_s\rangle$, Eq. (3.3), which enjoys the expansion coefficients

$$\psi_m = \begin{cases} (1+m)^{-1} & , \text{ for } 0 \leq m \leq m_o \\ 0 & , \text{ for } m > m_o . \end{cases}$$

The width $\Delta\phi [|\psi_s\rangle]$, Eq. (2.4), then reads

$$\Delta\phi^2 [|\psi_s\rangle] = \frac{\left(\sum_{m=0}^{m_o} \frac{1}{1+m} \right)^2}{\left(\sum_{m=0}^{m_o} \frac{m^2}{1+m} \right) \left(\sum_{m=0}^{m_o} \frac{1}{1+m} \right) - \left(\sum_{m=0}^{m_o} \frac{m}{1+m} \right)^2} .$$

When we use the summation formulae, (Ref. 11)

$$\begin{aligned} f(m_o) &\equiv \sum_{m=0}^{m_o} \frac{1}{1+m} \\ &= C + \ln(m_o + 1) + O\left[(m_o + 1)^{-1}\right], \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} C &\equiv \lim_{s \rightarrow \infty} \left(\sum_{m=1}^s \frac{1}{m} - \ln s \right) \\ &= 0.577215 \dots \end{aligned}$$

is Euler's constant, and from Eq. (3.11)

$$\begin{aligned} \sum_{m=0}^{m_o} (1+m) &= (m_o + 1) + \frac{1}{2} m_o (m_o + 1) \\ &= \frac{1}{2} (m_o + 1) (m_o + 2), \end{aligned}$$

we find

$$\begin{aligned} \Delta\phi^2 [|\psi_s\rangle] &= f^2(m_o) \left[f(m_o) \sum_{m=0}^{m_o} \frac{(m+1)^2 - 2(m+1) + 1}{m+1} - \left(\sum_{m=0}^{m_o} \frac{m+1-1}{m+1} \right)^2 \right]^{-1} \\ &= f^2(m_o) \left\{ \left[\sum_{m=0}^{m_o} (m+1) - 2(m_o + 1) + f(m_o) \right] f(m_o) - \left[(m_o + 1) - f(m_o) \right]^2 \right\}^{-1} \\ &= f^2(m_o) \left[\frac{1}{2} (m_o + 1) (m_o + 2) f(m_o) - (m_o + 1)^2 \right]^{-1} . \end{aligned}$$

In the large m_o limit we arrive at

$$\Delta\phi^2 \left[|\psi_s\rangle \right] = \frac{2f(m_o)}{(m_o+1)^2} \left[1 + \frac{1}{m_o+1} - \frac{2}{f(m_o)} \right]^{-1}$$

$$= \frac{2f(m_o)}{(m_o+1)^2} + O \left[(m_o+1)^{-3} \right]$$

and with Eq. (3.15) we then have

$$\Delta\phi^2 \left[|\psi_s\rangle \right] \cong \frac{2 \left[C + \ln(m_o+1) \right]}{(m_o+1)^2}. \quad (3.16)$$

The average number of photons in this state reads

$$\langle m+1 \rangle = \mathcal{X}^2 (m_o) \sum_{m=0}^{m_o} \frac{1}{m+1}$$

$$= \mathcal{X}^2 (m_o) \left[C + \ln(m_o+1) + \frac{1}{2(m_o+1)} + O \left(\frac{1}{m_o^2} \right) \right]$$

$$= \frac{6}{\pi^2} \left[C + \ln(m_o+1) \right] + O \left[\frac{\ln(m_o+1)}{m_o+1} \right],$$

(3.17a)

that is,

$$m_o+1 \cong \gamma^{-1} \exp \left[\frac{\pi^2}{6} \langle m+1 \rangle \right], \quad (3.17b)$$

where we have defined $\gamma \equiv e^C$ in a standard notation. In determining the remainder, we have applied in the last step of Eq. (3.17a) the asymptotic expression for \mathcal{X} , Eq. (3.5). With the help of Eqs. (3.17a) and (3.17b), Eq. (3.16) reads

$$\Delta\phi \left[|\psi_s\rangle \right] = \frac{\gamma\pi}{\sqrt{3}} \langle m+1 \rangle^{1/2} e^{-\frac{\pi^2}{6} \langle m+1 \rangle}$$

$$+ O \left[\langle m+1 \rangle^{-1/2} e^{-\frac{\pi^2}{6} \langle m+1 \rangle} \right], \quad (3.18)$$

which shows that the Gaussian-approximated width $\Delta\phi$ of the phase state $|\psi_s\rangle$ decreases *exponentially* with the average number of photons. The height

$$W_{\phi=0} = \left(2\pi \mathcal{X}^2 \right)^{-1} \left[\mathcal{X}^2 \sum_{m=0}^{m_o} (1+m)^{-1} \right]^2$$

$$= \left(2\pi \mathcal{X}^2 \right)^{-1} \langle m+1 \rangle^2$$

$$\cong \frac{\pi}{12} \langle m+1 \rangle^2 \quad (3.19)$$

of the distribution increases only *quadratically*. Here, we have also made use of Eq. (3.17a). Hence, the phase probability or area caught underneath the peak,

$$A_{\text{Gauss}} \left[|\psi_s\rangle \right] = \frac{\pi^{5/2} \gamma}{12\sqrt{3}} \langle m+1 \rangle^{5/2} e^{-\frac{\pi^2}{6} \langle m+1 \rangle} + O \left(\langle m+1 \rangle^{3/2} e^{-\frac{\pi^2}{6} \langle m+1 \rangle} \right),$$

decreases rapidly to zero with increasing average photon number. All the probability is stored in the broad wings of the distribution.

IV. SUMMARY AND OUTLOOK: A NEW VARIATIONAL PROBLEM

In this article we have presented a Gaussian approximation to the maximum of the semiclassical phase distribution of an arbitrary quantum state. We have illustrated this scheme using the example of a coherent state, a truncated phase state, and the intriguing optimal phase state. Our main results are summarized in Table 1. The Gaussian-approximated width $\Delta\phi[|\psi_{\text{coh}}\rangle]$, Eq. (3.7), of the coherent state $|\psi_{\text{coh}}\rangle$ decreases as the square root of the average number of photons, whereas for a truncated phase state $|\psi_p\rangle$ the width $\Delta\phi[|\psi_p\rangle]$, Eq. (3.12), decreases linearly with $\langle m \rangle$. In both cases the Gaussian is properly normalized, that is, the probability caught underneath the peak is almost

identical to unity, Eqs. (3.9) and (3.14), and independent of $\langle m \rangle$.

The situation is quite different for the optimal phase state $|\psi_s\rangle$. Here, $\Delta\phi[|\psi_s\rangle]$, Eq. (3.18), decreases exponentially with $\langle m \rangle$, but the maximum $W_{\phi=0}[|\psi_s\rangle]$, Eq. (3.19), increases only quadratically with $\langle m \rangle$, leading to vanishing probability in the peak. All probability in this case is in the broad wings of the distribution, as is discussed in detail in Ref. 9.

We conclude by noting that the Gaussian approximation might lead to insight into questions such as the existence of a lower bound of $\Delta\phi$ for a given fixed number of photons $\langle m \rangle$. For that purpose we would like all the probability to reside in the peak, that is, $A_{\text{Gauss}} = 1$. From Eq. (2.5), we find

$$W_{\phi=0} = \pi^{-1/2} \Delta\phi^{-1}$$

which, when substituted into Eq. (2.4b), yields

$$\begin{aligned} \Delta\phi^3 &= \frac{2}{\sqrt{\pi}} \left| W''_{\phi=0} \right|^{-1} = 2\sqrt{\pi} \chi^{-2} \left[\left(\sum_{m=0}^{\infty} m^2 \psi_m \right) \left(\sum_{m=0}^{\infty} \psi_m \right) - \left(\sum_{m=0}^{\infty} m \psi_m \right)^2 \right]^{-1} \\ &= 4\sqrt{\pi} \chi^{-2} \left[\sum_{m,n=0}^{\infty} (m-n)^2 \psi_m \psi_n \right]^{-1}. \end{aligned} \quad (4.1)$$

Two strategies offer themselves: (i) Use appropriate inequalities to rewrite the expression in square brackets in Eq. (4.1) in terms of the average number of photons and its variance. This might lead to a lower bound of $\Delta\phi$. (ii) Minimize $\Delta\phi^3$, that is, maximize the expression in square brackets, Eq. (4.1), under the constraint of constant $\langle m \rangle$ and phase state normalized to unity.

ACKNOWLEDGEMENTS

We would like to thank R. E. Slusher and B. Yurke for drawing this problem to our attention. In particular, we thank V. Akulin, R. Bruch, C. M. Caves, R. Hellwarth, and Y. Yamamoto for many fruitful discussions.

One of us (JPD) acknowledges H. Walther and the Max-Planck-Institut für Quantenoptik for hospitality and support. We also would like to express our deepest appreciation to D. Han, Y. S. Kim, and W. W. Zachary for organizing a most splendid conference.

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Table 1. Gaussian approximation $W_\varphi = W_{\varphi=0} \exp [-(\varphi/\Delta\varphi)^2]$ for dominant maximum of the phase distribution $W_\varphi = (2\pi)^{-1} \mathcal{N}^2 \left| \sum_{m=0}^{\infty} \psi_m \exp (im\varphi) \right|^2$ for a coherent state $|\psi_{\text{coh}}\rangle$, a truncated phase state $|\psi_p\rangle$ and an optimal phase state $|\psi_s\rangle$.

	Coherent State	Truncated Phase State	Optimal Phase State
Width = $\Delta\varphi$	$2^{-1/2} \langle m \rangle^{-1/2}$	$\sqrt{3} \langle m \rangle^{-1}$	$(\gamma\pi/\sqrt{3}) \langle m+1 \rangle^{1/2} \exp [-\pi^2 \langle m+1 \rangle/6]$
Maximum = $W_{\varphi=0}$	$\sqrt{2/\pi} \langle m \rangle^{1/2}$	$\pi^{-1} \langle m \rangle$	$\frac{\pi}{12} \langle m+1 \rangle^2$
Area = $\sqrt{4} \Delta\varphi W_{\varphi=0}$	1	$\sqrt{3/\pi} = 0.98$	$\frac{\pi}{12}^{5/2} \frac{\gamma}{\sqrt{3}} \langle m+1 \rangle^{5/2} \exp [-\pi^2 \langle m+1 \rangle/6]$ $\rightarrow 0$ $(m \rightarrow \infty)$